

# ON TWO PROBLEMS CONCERNING TOPOLOGICAL CENTERS

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**ABSTRACT.** Let  $\Gamma$  be an infinite discrete group and  $\beta\Gamma$  its Čech-Stone compactification. Using the well known fact that a free ultrafilter on an infinite set is nonmeasurable, we show that for each element  $p$  of the remainder  $\beta\Gamma \setminus \Gamma$ , left multiplication  $L_p : \beta\Gamma \rightarrow \beta\Gamma$  is not Borel measurable. Next assume that  $\Gamma$  is abelian. Let  $\mathcal{D} \subset \ell^\infty(\Gamma)$  denote the subalgebra of distal functions on  $\Gamma$  and let  $D = \Gamma^{\mathcal{D}} = |\mathcal{D}|$  denote the corresponding universal distal (right topological group) compactification of  $\Gamma$ . Our second result is that the topological center of  $D$  (i.e. the set of  $p \in D$  for which  $L_p : D \rightarrow D$  is a continuous map) is the same as the algebraic center and that for  $\Gamma = \mathbb{Z}$ , this center coincides with the canonical image of  $\Gamma$  in  $D$ .

## 1. INTRODUCTION

This short note is a direct outcome of the topology conference held in Castellón in the summer of 2007. I was presented during the conference with two problems relating to the topological center of certain right topological semigroups. (A compact semigroup  $A$  such that for each  $p \in A$  the corresponding right multiplication  $R_p : q \mapsto qp$  is continuous is called a *right topological semigroup*. The collection of elements  $p \in A$  for which the corresponding left multiplication  $L_p : q \mapsto pq$  is continuous is called the *topological center* of  $A$ .) The first was a question of Michael Megrelishvili: Given an infinite discrete group  $\Gamma$ , which are the elements of  $\beta\Gamma$  for which  $L_p : \beta\Gamma \rightarrow \beta\Gamma$  is a Baire class 1 map? (It is known that the topological center of  $\beta\Gamma$  is exactly  $\Gamma$  itself, considered as a subset of  $\beta\Gamma$ , see e.g. [3].) The second problem is due to Mahmoud Filali: If  $D = D(\Gamma)$  is the universal distal Ellis group of  $\Gamma$ , identify the topological center of  $D$ .

I present here a complete answer to Megrelishvili's problem, based on the well known result that a free ultrafilter on an infinite set is nonmeasurable, and an answer to Filali's problem in the case  $\Gamma = \mathbb{Z}$ , the group of integers.

The interested reader is referred to [1, chapter 1] and the bibliography list thereof, for more information on the abstract theory of topological dynamics, and to [3] for information concerning  $\beta\Gamma$ .

I thank both Megrelishvili and Filali for addressing to me these nice problems. I also thank the organizers of the Castellón meeting for the formidable effort they put into the details of the conference and for their warm hospitality.

## 2. ON THE CENTER OF $\beta\Gamma$

**Theorem 2.1.** *Let  $\Gamma$  be an infinite discrete group and  $\beta\Gamma$  its Čech-Stone compactification. For each element  $p$  of the remainder  $\beta\Gamma \setminus \Gamma$ , left multiplication  $L_p : \beta\Gamma \rightarrow \beta\Gamma$  is not Borel measurable.*

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*Proof.* Let  $\mathcal{P}(\Gamma)$  denote the collection of all subsets of  $\Gamma$ . Let  $\Omega = \{0, 1\}^\Gamma$  and let  $\chi : \mathcal{P}(\Gamma) \rightarrow \Omega$  denote the canonical map  $\chi(A) = \mathbf{1}_A$  for  $A \subset \Gamma$ . We regard  $\Omega$  as a compact space and let  $\mathcal{B}$  denote its Borel  $\sigma$ -algebra. Let  $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^\Gamma$  denote the product probability measure on  $\Omega$  and let  $\mathcal{B}_\mu$  denote the completion of  $\mathcal{B}$  with respect to  $\mu$ .

A well known and easy fact, which for completeness we will reproduce below (Lemma 2.3), is that a free ultrafilter on an infinite set is nonmeasurable: Viewing an element  $p \in \beta\Gamma \setminus \Gamma$  as an ultrafilter on  $\Gamma$ , the collection  $\{\chi(A) : A \in p\} \subset \Omega$ , is not  $\mu$ -measurable; i.e. not an element of  $\mathcal{B}_\mu$ . In particular it is not a Borel subset of  $\Omega$ . (In fact, it is not even Baire measurable, [4] and [5].)

The compact space  $\Omega$  becomes a dynamical system when we let  $\Gamma$  act on it by permuting the coordinates:

$$\gamma\omega(\gamma') = \omega(\gamma^{-1}\gamma').$$

Of course the measure  $\mu$  is  $\Gamma$ -invariant, but we will not need this fact. The action of  $\Gamma$  on  $\Omega$  extends to an action of  $\beta\Gamma$  in the natural way and we write  $p\omega$  for the image of  $\omega \in \Omega$  under  $p \in \beta\Gamma$ . (In fact via this “action”  $\beta\Gamma$  is identified with the enveloping semigroup of the system  $(\Omega, \Gamma)$ , see [1, chapter 1].)

For  $A \subset \Gamma$  and  $p \in \beta\Gamma$  set

$$p \star A = \{\gamma \in \Gamma : \gamma A^{-1} \in p\}$$

and check that  $\gamma\mathbf{1}_A = \mathbf{1}_{\gamma A}$  and  $p\chi(A) = p\mathbf{1}_A = \mathbf{1}_{p \star A} = \chi(p \star A)$ . Moreover if  $q \in \beta\Gamma$  then

$$pq \star A = p \star (q \star A).$$

For convenience I sometimes identify a subset  $A \subset \Gamma$  with the corresponding element  $\mathbf{1}_A = \chi(A)$  in  $\Omega$ .

Let  $\pi_e : \Omega \rightarrow \{0, 1\}$  denote the projection on the  $e$ -component of  $\Omega$ . Here  $e$  is the neutral element of  $\Gamma$ . Let  $\omega_0 = \mathbf{1}_D$  be some fixed element of  $\Omega$  whose  $\Gamma$  orbit is dense in  $\Omega$ . Let  $\psi : \gamma \mapsto \gamma\omega_0$  be the orbit map and let  $\hat{\psi}$  denote its unique extension to  $\beta\Gamma$ . Thus  $\hat{\psi}(q) = q\omega_0$  for  $q \in \beta\Gamma$ . Finally recall that the semigroup product on  $\beta\Gamma$  is defined by

$$A \in pq \iff \{\gamma \in \Gamma : \gamma^{-1}A \in q\} \in p.$$

Consider the map  $L_p : \beta\Gamma \rightarrow \beta\Gamma$  and write  $\phi : \beta\Gamma \rightarrow \{0, 1\}$  for the map  $\phi = \pi_0 \circ \hat{\psi} \circ L_p$ . (Thus  $\phi(q) = (pq\omega_0)(e)$ .) We define  $J : \Omega \rightarrow \Omega$  by  $(J(\omega))(\gamma) = \omega(\gamma^{-1})$

We have

$$\mathcal{Q} := \phi^{-1}(1) = \{q \in \beta\Gamma : pq\omega_0(e) = 1\}.$$

Now  $pq\omega_0(e) = 1$  iff  $e \in \chi^{-1}(pq\omega_0)$  hence

$$\begin{aligned} \mathcal{Q} &= \{q \in \beta\Gamma : pq\omega_0(e) = 1\} \\ &= \{q \in \beta\Gamma : e \in p \star q\omega_0\} \\ &= \{q \in \beta\Gamma : e \in p \star q \star D\} \\ &= \{q \in \beta\Gamma : e \in \{\gamma \in \Gamma : \gamma(q \star D)^{-1} \in p\}\} \\ &= \{q \in \beta\Gamma : (q \star D)^{-1} \in p\} \\ &= \{q \in \beta\Gamma : J(q\omega_0) \in p\}. \end{aligned}$$

Thus  $(J \circ \hat{\psi})(\mathcal{Q}) = p$  and since also  $(\hat{\psi}^{-1} \circ J^{-1})(p) = \mathcal{Q}$  we conclude that  $\mathcal{Q} = \phi^{-1}(1)$  is not Borel measurable in  $\beta\Gamma$ . Finally, since also  $\mathcal{Q} = L_p^{-1}(\{q \in \beta\Gamma : (q\omega_0)(e) = 1\})$  we see that  $L_p$  is not Borel measurable.  $\square$

In the next two lemmas let  $\Omega = \{0, 1\}^\mathbb{N}$ . As above, we identify subsets  $A$  of  $\mathbb{N}$  with their characteristic functions  $\mathbf{1}_A \in \Omega = \{0, 1\}^\mathbb{N}$  and, accordingly, filters on  $\mathbb{N}$  with subsets of  $\Omega$ .

Let  $\phi : \Omega \rightarrow \Omega$  denote the “flip” function defined by  $\phi(\omega)_n = 1 - \omega_n$ . We consider the measure space  $(\Omega, \Sigma_\lambda, \lambda)$ , where  $\Omega = \{0, 1\}^\mathbb{N}$ ,  $\lambda$  is the Bernoulli measure  $\lambda = (\frac{1}{2}(\delta_0 + \delta_1))^\mathbb{N}$ , and  $\Sigma_\lambda$  denotes the completion of the Borel  $\sigma$ -algebra with respect to  $\lambda$ . As usual we use the notation  $\lambda_*$  and  $\lambda^*$  for the induced inner and outer measures.

The assertions of the next lemma are easily verified.

- Lemma 2.2.** 1. *The involution  $\phi$  is measurable and it preserves  $\lambda$ .*  
 2. *For  $A \subset \mathbb{N}$  we have  $\phi(\mathbf{1}_A) = \mathbf{1}_{A^c}$ .*  
 3. *If  $\mathcal{F}$  is a filter on  $\mathbb{N}$  then  $\phi\mathcal{F} \cap \mathcal{F} = \emptyset$ .*  
 4. *If  $\mathcal{F}$  is a free filter on  $\mathbb{N}$  (i.e.  $\bigcap \mathcal{F} = \emptyset$ ) then, considered as a collection of subsets of  $\{0, 1\}^\mathbb{N}$  it is a “tail event”, that is, for every  $m \in \mathbb{N}$ ,  $\mathcal{F} = \{0, 1\}^m \times \mathcal{F}'$ , with  $\mathcal{F}' \subset \{0, 1\}^\mathbb{N}$ .*  
 5. *A filter  $\mathcal{F}$  on  $\mathbb{N}$  is an ultrafilter iff  $\phi(\mathcal{F}) \cup \mathcal{F} = \Omega$ .*

**Lemma 2.3.** *Let  $\mathcal{F}$  be a free filter on  $\mathbb{N}$ . Then*

1.  $\lambda_*(\mathcal{F}) = 0$ .
2.  $\lambda^*(\mathcal{F}) \in \{0, 1\}$ .
3.  $\lambda^*(\mathcal{F}) = 1$  if  $\mathcal{F}$  is an ultrafilter.
4. *A free filter  $\mathcal{F}$  is measurable iff  $\lambda^*(\mathcal{F}) = 0$ , and nonmeasurable iff  $\lambda^*(\mathcal{F}) = 1$ . In particular, every free ultrafilter is nonmeasurable.*

*Proof.* If  $\mathcal{F}$  is a free filter on  $\mathbb{N}$  and  $\mathcal{G} \subset \mathcal{F}$  is a measurable tail event then it has measure either 0 or 1. Thus  $\lambda^*(\mathcal{F}) \in \{0, 1\}$ . This proves part 2. We also have  $\lambda_*(\mathcal{F}) \in \{0, 1\}$  and since  $\phi(\mathcal{F}) \cap \mathcal{F} = \emptyset$  it follows that

$$1 = \lambda(\Omega) \geq \lambda_*(\phi\mathcal{F}) + \lambda_*(\mathcal{F}) = 2\lambda_*(\mathcal{F}).$$

We conclude that  $\lambda_*(\mathcal{F}) = 0$ , proving part 1. If  $\mathcal{F}$  is an ultrafilter then  $\mathcal{F} \cup \phi\mathcal{F} = \{0, 1\}^\mathbb{N}$  and we conclude that

$$1 = \lambda(\Omega) \leq \lambda^*(\phi\mathcal{F}) + \lambda^*(\mathcal{F}) = 2\lambda^*(\mathcal{F}),$$

whence  $\lambda^*(\mathcal{F}) = 1$ . This proves part 3. Part 4 is now clear.  $\square$

### 3. ON THE CENTER OF $\Gamma^\mathcal{D}$ , THE UNIVERSAL DISTAL ELLIS GROUP OF $\Gamma$

Let  $\Gamma$  be a discrete abelian group. Let  $\mathcal{D}$  denote the closed  $\Gamma$ -invariant subalgebra of (complex valued) distal functions in  $\ell^\infty(\Gamma)$ . Let  $D = \Gamma^\mathcal{D} = |\mathcal{D}|$  denote the corresponding Gelfand space. It is well known that  $D$  is the largest right topological group compactification of  $\Gamma$ .

**Theorem 3.1.** *Let  $\Gamma$  be an infinite discrete abelian group. The topological center of  $D = \Gamma^\mathcal{D}$  is the same as the algebraic center and, when  $\Gamma = \mathbb{Z}$ , it also coincides with the canonical image of  $\Gamma$  in  $D$ .*

*Proof.* In order to simplify our notation we will identify elements of  $\Gamma$  with their images in  $D$ . The coincidence of the topological and algebraic centers of  $D$  is easy: Suppose first that  $p \in D$  is in the algebraic center of this group. Then, as right multiplication is always continuous, we have for any convergent net  $q_\alpha \rightarrow q$  in  $D$

$$pq = qp = \lim q_\alpha p = \lim pq_\alpha,$$

i.e.  $L_p : D \rightarrow D$  is continuous.

Conversely, assume that  $p$  is in the topological center; i.e.  $L_p : D \rightarrow D$  is continuous. We note that if  $q$  is an element of  $D$  then  $\gamma q = q\gamma$  for every  $\gamma \in G$ . In fact choosing a convergent net  $\Gamma \ni \gamma_\alpha \rightarrow q$ , by the commutativity of  $\Gamma$ ,

$$\gamma q = \gamma \lim \gamma_\alpha = \lim \gamma \gamma_\alpha = \lim \gamma_\alpha \gamma = q\gamma.$$

Now, with this in mind, we have

$$pq = p \lim \gamma_\alpha = \lim p\gamma_\alpha = \lim \gamma_\alpha p = qp,$$

so that  $p$  is indeed an element of the center.

Now to the more delicate task of showing that this center coincides with  $\Gamma$ . Let  $p \in D$  be a central element. If  $p \notin \Gamma$  then there exists a *metric* minimal distal dynamical system  $(Y, \Gamma)$  and a point  $y_0 \in Y$  such that

$$(3.1) \quad py_0 \notin \Gamma y_0.$$

By assumption the map  $L_p : D \rightarrow D$  is continuous (in fact a homeomorphism) and as we have seen it also commutes with every element of  $\Gamma$ . In other words,  $L_p$  is an automorphism of the system  $(D, \Gamma)$ . Now the dynamical system  $(D, \Gamma)$  is the universal distal system and therefore, it admits a unique homomorphism of dynamical systems  $\hat{\phi} : (D, \Gamma) \rightarrow (E(Y, \Gamma), \Gamma)$  onto the enveloping semigroup  $E = E(Y, \Gamma)$  (which by a theorem of Ellis is in fact a group) such that  $\hat{\phi}(e_D) = e_E$ . Now the map  $\phi : p \mapsto \hat{\phi}(p)y_0$  (which we write simply as  $p \mapsto py_0$ ) is a homomorphism  $\phi : (D, \Gamma) \rightarrow (Y, \Gamma)$  with  $\phi(e) = y_0$ . If  $y_\alpha \rightarrow y$  is a convergent net in  $Y$  then there are  $q_\alpha \in D$  with  $y_\alpha = q_\alpha y_0$ . With no loss of generality we have  $q_\alpha \rightarrow q$  in  $D$ , so that in particular  $y = \lim y_\alpha = \lim q_\alpha y_0 = qy_0$ . Now we see that

$$\begin{aligned} py &= pqy_0 = (p \lim q_\alpha)y_0 \\ &= (\lim pq_\alpha)y_0 = \lim p(q_\alpha y_0) \\ &= \lim py_\alpha. \end{aligned}$$

Thus  $p$  acts continuously on  $Y$ . Since also  $p\gamma = \gamma p$  for every  $\gamma \in \Gamma$  we conclude that  $p$  is an automorphism of the system  $(Y, \Gamma)$ .

Note that this argument shows that  $p$  acts as an automorphism of every factor of  $(D, \Gamma)$ . Therefore, our proof will be complete when we find a minimal distal dynamical system  $(X, \Gamma)$  extending  $(Y, \Gamma)$ , say  $\pi : (X, \Gamma) \rightarrow (Y, \Gamma)$ , where  $p$  is not an automorphism.

At this stage, in order to be able to use a method of construction developed by Glasner and Weiss in [2], we specialize to the case  $\Gamma = \mathbb{Z}$ . In particular the system  $(Y, \Gamma)$  which was singled out in the above discussion has now the form  $(Y, T)$  where  $T : Y \rightarrow Y$  is a self homeomorphism of  $Y$  determined by the element  $1 \in \mathbb{Z}$ . Of course we can assume that  $Y$  is non-periodic (i.e. infinite).

The following construction is a special case of a general setup designed in [2] for providing minimal extensions of a given non-periodic minimal  $\mathbb{Z}$ -system  $(Y, T)$ . We refer the reader to [2] for more details.

Set  $X = Y \times K$  where  $K$  denotes the circle group  $K = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . Let  $\Theta$  be the family of continuous maps  $\theta : Y \rightarrow K$ . For each  $\theta \in \Theta$  let  $G_\theta : X \rightarrow X$  be the map  $G_\theta(y, z) = (y, z\theta(y))$  and  $S_\theta = G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta$ . Thus

$$(3.2) \quad S_\theta : X \rightarrow X, \quad S_\theta(y, z) = (Ty, z\theta(y)\theta(Ty)^{-1}).$$

Form the collection

$$\mathcal{S}(T) = \{G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta : \theta \in \Theta\}.$$

Theorem 1 of [2] ensures that in the set  $\overline{\mathcal{S}(T)}$  (closure with respect to the uniform convergence topology in  $\text{Homeo}(X)$ ) there is a dense  $G_\delta$  subset  $\mathcal{R}$  such that for every  $R \in \mathcal{R}$

the system  $(X, R)$  is minimal, distal, and the projection map  $\pi : X \rightarrow Y$  is a homomorphism of dynamical systems ( $\pi R(y, z) = T\pi(y, z) = Ty$ ). Note that every  $R \in \overline{\mathcal{S}(T)}$  has the form

$$R = T_\phi : X \rightarrow X, \quad \text{where} \quad T_\phi(y, z) = (Ty, z\phi(y)),$$

for some continuous map  $\phi : Y \rightarrow K$ . We will often use the fact that for  $n \in \mathbb{N}$  the  $n$ -th iteration of  $T_\phi$  has the form

$$(3.3) \quad T_\phi^n(y, z) = (T^n y, z\phi_n(y)), \quad \text{where} \quad \phi_n(y) = \phi(T^{n-1}y) \cdots \phi(Ty)\phi(y).$$

Note that when  $\phi$  has the very special form  $\phi(y) = \theta(Ty)^{-1}\theta(y)$  for some continuous  $\theta : Y \rightarrow K$ , the equation (3.3) collapses:

$$(3.4) \quad \phi_n(y) = \theta(T^n y)^{-1}\theta(y), \quad \text{hence} \quad S_\theta^n(y, z) = (T^n y, z\theta(T^n y)^{-1}\theta(y)).$$

We temporarily fix an element  $R = T_\phi \in \mathcal{R}$ . As observed above, the element  $p \in D$  defines an automorphism of the system  $(X, T_\phi)$ ; moreover we have for every  $x = (y, z) \in X$ :

$$\pi(px) = p\pi(x) = py.$$

This last observation implies that  $p : X \rightarrow X$  has the form  $p(y, z) = (py, \omega(y, z))$  for some continuous map  $\omega : Y \times K \rightarrow K$ .

**Lemma 3.2.** *The function  $\omega$  has the form  $\omega(y, z) = z\psi(y)$  for some continuous map  $\psi : Y \rightarrow K$ , whence*

$$p(y, z) = (py, z\psi(y))$$

*Proof.* There exists a net  $\{n_\nu\}_{\nu \in I}$  in  $\mathbb{Z}$  such that  $p = \lim n_\nu$  in  $D$ . Thus, for every  $(y, z) \in X$

$$p(y, z) = \lim T_\phi^{n_\nu}(y, z) = \lim(T_\phi^{n_\nu}y, z\phi_{n_\nu}(y)) = (py, z\psi(y)),$$

where the *point-wise* limit  $\psi(y) := \lim \phi_{n_\nu}(y)$  is necessarily a continuous function.  $\square$

The commutation relation  $pT_\phi = T_\phi p$  now reads:

$$\begin{aligned} pT_\phi(y, z) &= p(Ty, z\phi(y)) = (pTy, z\phi(y)\psi(Ty)) \\ &= T_\phi p(y, z) = T_\phi(py, z\psi(y)) \\ &= (Tpy, z\psi(y)\phi(py)). \end{aligned}$$

In turn this implies:

$$(3.5) \quad \phi(y)\psi(Ty) = \psi(y)\phi(py).$$

Similarly the commutation relations  $pT_\phi^n = T_\phi^n p$  yield:

$$(3.6) \quad \phi_n(y)\psi(T^n y) = \psi(y)\phi_n(py).$$

Next consider any sequence  $n_i \nearrow \infty$  such that

- $\lim T^{n_i} y_0 = y_0$ ,
- $\lim \phi_{n_i}(y_0) = z'$ , and
- $\lim \phi_{n_i}(py_0) = z''$ .

Applying (3.6) and taking the limit as  $i \rightarrow \infty$  we get  $\psi(y_0)z'' = z'\psi(y_0)$ , whence necessarily also  $z' = z''$ .

The proof of Theorem 3.1 will be complete when we next show that for a residual subset  $\mathcal{R}_1$  of  $\overline{\mathcal{S}(T)}$ , we have  $\lim \phi_{n_i}(y_0) = z' \neq z'' = \lim \phi_{n_i}(py_0)$ , whenever  $R = T_\phi \in \mathcal{R}_1$ . Then for any element  $T_\phi \in \mathcal{R} \cap \mathcal{R}_1$ ,  $(X, T_\phi)$  will serve as a minimal distal system where  $p$  is not an automorphism.

**Proposition 3.3.** *For a given sequence  $n_i \nearrow \infty$  with  $\lim T^{n_i} y_0 = y_0$ , the set*

$$\mathcal{R}_1 = \{T_\phi \in \overline{\mathcal{S}(T)} : \forall i \exists j > i, |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1\}$$

*is a residual subset of  $\overline{\mathcal{S}(T)}$ .*

*Proof.* For  $i \in \mathbb{N}$  and  $\eta > 0$  set

$$E_{i,\eta} = \{T_\phi \in \overline{\mathcal{S}(T)} : \exists j > i, |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta\}.$$

Clearly  $E_{i,\eta}$  is an open subset of  $\overline{\mathcal{S}(T)}$  and for  $i < k$  we have  $E_{k,\eta} \subset E_{i,\eta}$ .

**Lemma 3.4.** *Given  $i$  and  $\eta > 0$ , for every  $\theta_0 \in \Theta$  there exists an  $i_0 > i$  such that*

$$G_{\theta_0}^{-1} E_{i_0,\eta} G_{\theta_0} \subset E_{i_0,\eta/2}.$$

*Proof.* Fix  $\theta_0 \in \Theta$ . For sufficiently large  $i_0$ , for all  $j > i_0$  the distances  $d(T^{n_j} y_0, y_0)$  and  $d(T^{n_j} py_0, py_0)$  are so small that

$$|\theta(T^{n_j} y_0)^{-1} \theta(y_0) \phi_{n_j}(y_0) - \theta(T^{n_j} py_0)^{-1} \theta(y_0) \phi_{n_j}(py_0)| > 1 + \eta/2$$

holds whenever

$$|\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta.$$

□

We will show that  $E_{i,\eta}$  is also dense in  $\overline{\mathcal{S}(T)}$ . For this it suffices to show that  $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in \overline{E_{i,\eta}}$  for every  $\theta \in \Theta$ , i.e.  $T \times \text{id} \in G_\theta \overline{E_{i,\eta}} G_\theta^{-1}$ .

Now for a fixed  $\theta_0$  there is by Lemma 3.4, an  $i_0 > i$  with  $G_{\theta_0}^{-1} E_{i_0,2\eta} G_{\theta_0} \subset E_{i_0,\eta}$ , hence it suffices to show that  $T \times \text{id} \in \overline{E_{i_0,2\eta}}$ , since then

$$T \times \text{id} \in \overline{E_{i_0,2\eta}} \subset G_{\theta_0} \overline{E_{i_0,\eta}} G_{\theta_0}^{-1} \subset G_{\theta_0} \overline{E_{i,\eta}} G_{\theta_0}^{-1}.$$

Finally the next lemma will prove this last assertion and therefore also the density of  $E_{i,\eta}$  for every  $i$  and  $0 < \eta < 1$ .

**Lemma 3.5.** *Given  $i \in \mathbb{N}$ ,  $0 < \eta < 1$  and  $\varepsilon > 0$  there exists  $\theta \in \Theta$  such that*

1.  $d(T \times \text{id}, G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta) < \varepsilon$ .
2.  $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in E_{i,\eta}$ .

*Proof.* Let  $I = [0, 1]$  and set  $h(0) = h(1/3) = h(2/3) = 1$ ,  $h(1) = -1$  and extend this function in an arbitrary way to a continuous  $h : I \rightarrow S^1$ . Choose  $\delta > 0$  such that  $|t - s| < \delta$  implies  $|h(t)^{-1} h(s) - 1| < \varepsilon$ . Let  $m \in \mathbb{N}$  be such that  $2/m < \delta$ . Let  $U_1$  and  $U_2$  be open neighborhoods of  $y_0$  and  $py_0$ , respectively, in  $Y$  such that for  $s = 1, 2$ , the sets  $U_s, TU_s, \dots, T^{m-1}U_s$  are mutually disjoint. (Here we use the facts that  $Y$  is infinite and that  $py_0 \notin \{T^j y_0 : j \in \mathbb{Z}\}$  (3.1).) Choose  $k > i$  so that  $T^{n_k} y_0 \in U_1$  and  $T^{n_k} py_0 \in U_2$ . Let  $K_s \subset U_s$ ,  $s = 1, 2$ , be Cantor sets such that  $y_0, T^{n_k} y_0 \in K_1$  and  $py_0, T^{n_k} py_0 \in K_2$ .

Next define:

$$g(y_0) = 0, \quad g(T^{n_k} y_0) = 1/3, \quad g(py_0) = 2/3, \quad g(T^{n_k} py_0) = 1$$

and extend this function in an arbitrary way to a continuous function  $g : K_1 \cup K_2 \rightarrow S^1$ . We now extend  $g$  to the set  $\cup_{j=0}^{m-1} T^j(K_1 \cup K_2)$  by setting  $g(y) = g(T^{-j} y)$  for  $y \in T^j(K_1 \cup K_2)$ . Extend  $g$  continuously over all of  $Y$  in an arbitrary way.

Set

$$\tilde{g}(y) = \frac{1}{m} \sum_{j=0}^{m-1} g(T^j y).$$

Clearly  $\tilde{g} \upharpoonright (K_1 \cup K_2) = g \upharpoonright (K_1 \cup K_2)$ , so that

$$\tilde{g}(y_0) = 0, \quad \tilde{g}(T^{n_k} y_0) = 1/3, \quad \tilde{g}(py_0) = 2/3, \quad \tilde{g}(T^{n_k} py_0) = 1.$$

Finally define  $\theta : Y \rightarrow S^1$  by  $\theta(y) = h(\tilde{g}(y))$ . Note that

$$(3.7) \quad \theta(y_0) = \theta(T^{n_k}y_0) = \theta(py_0) = 1, \quad \text{and} \quad \theta(T^{n_k}py_0) = -1.$$

Now

$$G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta(y, z) = (Ty, z\theta(Ty)^{-1}\theta(y)) = (Ty, zh(\tilde{g}(Ty))^{-1}h(\tilde{g}(y))).$$

But

$$|\tilde{g}(Ty) - \tilde{g}(y)| < 2/m < \delta,$$

hence  $|h(\tilde{g}(Ty))^{-1}h(\tilde{g}(y)) - 1| < \varepsilon$  and therefore also

$$d(T \times \text{id}, G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta) < \varepsilon.$$

This proves part (1) of the lemma and we now proceed to prove part (2). We have to show that  $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in E_i$ . But this map has the form

$$S_\theta : X \rightarrow X, \quad S_\theta(y, z) = G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta = (Ty, z\theta(Ty)^{-1}\theta(y)),$$

so that, by (3.4), we have to show that there exists  $j > i$  with

$$|\theta(T^{n_j}y_0)^{-1}\theta(y_0) - \theta(T^{n_j}py_0)^{-1}\theta(py_0)| > 1 + \eta.$$

Since, by the choice of  $\theta$  (3.7), we have

$$|\theta(T^{n_k}y_0)^{-1}\theta(y_0) - \theta(T^{n_k}py_0)^{-1}\theta(py_0)| = |1 - (-1)| = 2 > 1 + \eta,$$

this completes the proof of the lemma. □

To conclude the proof of Proposition 3.3 observe that, for instance, the dense  $G_\delta$  set  $\bigcap_{i=1}^\infty E_{i,1/2}$  is contained in  $\mathcal{R}_1$ . □

This also concludes the proof of Theorem 3.1. □

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